Division of Strength of Materials and Structures
Faculty of Power and Aeronautical Engineering

Finite element method (FEM1)

Lecture 6B. Requirements for the shape functions

03.2026

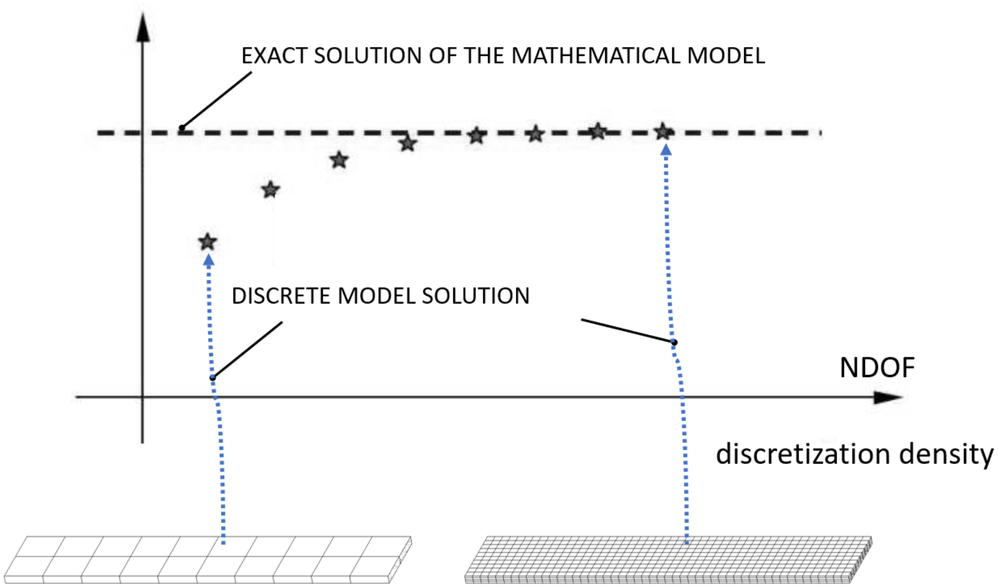
Requirements for the shape functions

a) The shape functions should allow for approximating a constant value of the function $\{u\}$ inside the finite element

b) The shape functions ensure the continuity, at the boundary between elements, of the displacement function $\{u\}$ and its derivatives to a degree one lower than the highest derivative of $\{u\}$ appearing in the total potential energy functional V.

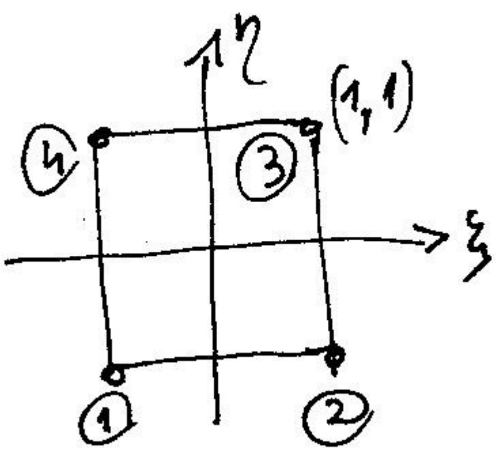
If requirements a) and b) are satisfied, the approximate solution tends to the exact solution with increasing the number of degrees of freedom.

Discrete model



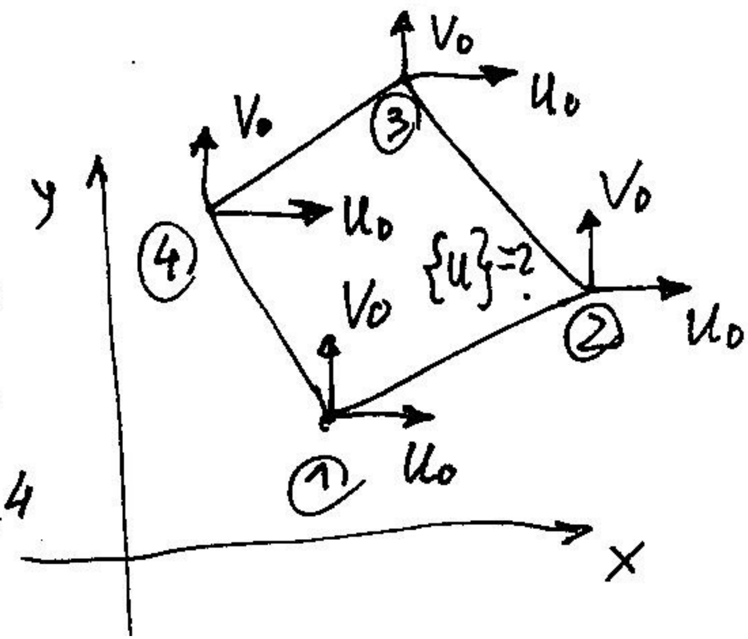
Example Check the requirements for the 4-node element shape functions

$$N_1(\xi,\eta) = \tfrac{1}{4}(1-\xi)(1-\eta)$$
$$N_2(\xi,\eta) = \tfrac{1}{4}(1+\xi)(1-\eta)$$
$$N_3(\xi,\eta) = \tfrac{1}{4}(1+\xi)(1+\eta)$$
$$N_4(\xi,\eta) = \tfrac{1}{4}(1-\xi)(1+\eta)$$



a)

$$u_i = u_0$$
$$v_i = v_0$$
$$i = 1, 2, 3, 4$$



$$\underset{2\times 1}{\{u\}} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \cdot \begin{Bmatrix} u_0 \\ v_0 \\ u_0 \\ v_0 \\ u_0 \\ v_0 \\ u_0 \\ v_0 \end{Bmatrix} =$$

$$= \begin{Bmatrix} (N_1 + N_2 + N_3 + N_4)\, u_0 \\ (N_1 + N_2 + N_3 + N_4)\, v_0 \end{Bmatrix} =$$

$$= \begin{Bmatrix} \frac{1}{4}\left((1-\xi)\cdot(1-\eta) + (1+\xi)(1-\eta) + (1+\xi)(1+\eta) + (1-\xi)(1+\eta) \right) \cdot u_0 \\ \frac{1}{4}(\ldots)\, v_0 \end{Bmatrix} =$$

$$= \begin{Bmatrix} \frac{1}{4}\left((1-\xi+1+\xi)(1-\eta) + (1+\xi+1-\xi)(1+\eta) \right) u_0 \\ \frac{1}{4}(\ldots)\, v_0 \end{Bmatrix} =$$

$$= \begin{Bmatrix} \frac{1}{4}\left(2(1-\eta) + 2(1+\eta) \right) u_0 \\ \frac{1}{4}(\ldots)\, v_0 \end{Bmatrix} = \begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix}$$

Condition a)
is met

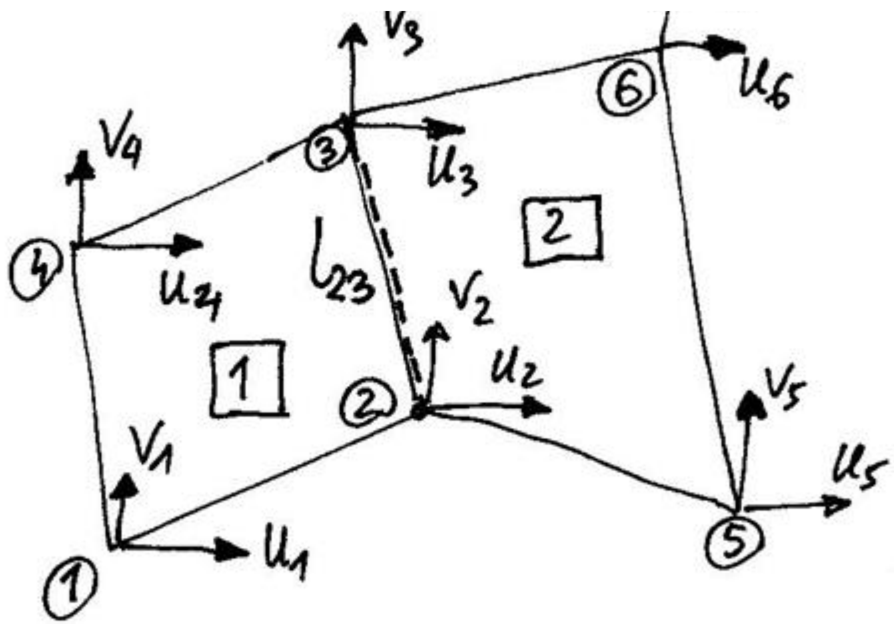
$$V_e = U_e - W_e = \frac{1}{2}\int_{\Omega_e} \underset{1\times 3}{\lfloor \varepsilon \rfloor}\,\underset{3\times 3}{[D]}\,\underset{3\times 1}{\{\varepsilon\}}\,d\Omega_e - \int_{\Omega_e} \underset{1\times 2}{\lfloor X \rfloor}\,\underset{2\times 1}{\{u\}}\,d\Omega_e - \int_{\Gamma_{pe}} \underset{1\times 2}{\lfloor p \rfloor}\,\underset{2\times 1}{\{u\}}\,d\Gamma_{pe}$$

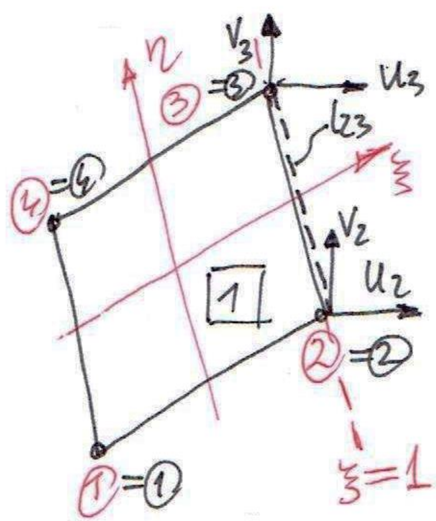
$$\underset{3\times 1}{\{\varepsilon\}} = \underset{3\times 2}{[R]}\,\underset{2\times 1}{\{u\}}$$

(The first order is the highest order of the derivative in the functional V)

contains differential operators of the first order

Condition b) is satisfied if the function $\{u\}$ is continuous between elements





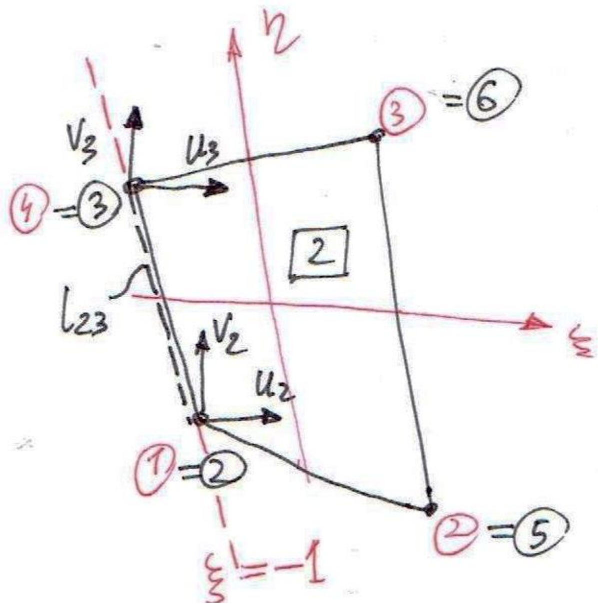
shape functions on the edge l_{23}

$$N_1 = 0,\ N_2 = \tfrac{1}{2}(1-\eta),\ N_3 = \tfrac{1}{2}(1+\eta),\ N_4 = 0$$

$$u\Big|_{23}^{\boxed{1}} = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4 =$$

$$= N_2 u_2 + N_3 u_3 = \tfrac{1}{2}\big((1-\eta)u_2 + (1+\eta)u_3\big)$$

$$v\Big|_{23}^{\boxed{1}} = \tfrac{1}{2}\big((1-\eta)v_2 + (1+\eta)v_3\big)$$



shape functions on the edge l_{23}

$$N_1 = \tfrac{1}{2}(1-\eta),\ N_2 = 0,\ N_3 = 0,\ N_4 = \tfrac{1}{2}(1+\eta)$$

$$u\Big|_{23}^{\boxed{2}} = N_1 \cdot u_2 + N_2 \cdot u_5 + N_3 \cdot u_6 + N_4 \cdot u_3 =$$

$$= N_1 \cdot u_2 + N_4 \cdot u_3 = \tfrac{1}{2}\big((1-\eta)u_2 + (1+\eta)u_3\big)$$

$$v\Big|_{23}^{\boxed{2}} = \tfrac{1}{2}\big((1-\eta)v_2 + (1+\eta)v_3\big)$$

$$u\Big|_{23}^{\boxed{1}} = u\Big|_{23}^{\boxed{2}} \quad \text{i} \quad v\Big|_{23}^{\boxed{1}} = v\Big|_{23}^{\boxed{2}} \Rightarrow$$

Condition b) is met